Reliable Computing **7:** 113–127, 2001.

On the Algebraic Properties of Intervals and Some Applications

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(Received: 3 May 1999; accepted: 10 October 1999)

Abstract. The algebraic properties of interval vectors (boxes) are studied. Quasilinear spaces with group structure are studied. Some fundamental algebraic properties are developed, especially in relation to the quasidistributive law, leading to a generalization of the familiar theory of linear spaces. In particular, linear dependence and basis are defined. It is proved that a quasilinear space with group structure is a direct sum of a linear and a symmetric space. A detailed characterization of symmetric quasilinear spaces with group structure is found.

1. Introduction

Many-dimensional intervals (interval vectors, boxes) have been increasingly used in interval analysis and reliable computing. Therefore it is necessary to study their algebraic properties and the operations and relations between them. Interval vectors form a quasilinear space with respect to addition and multiplication by scalar in the sense of [13]. A quasilinear space is an abelian cancellative monoid with respect to addition, in particular, the monoid can be a group. By means of the familiar extension method (used, e. g., when defining negative numbers) any quasilinear space can be embedded into a quasilinear space which is a group [7]. Therefore it is important to study quasilinear spaces with group structure. Quasilinear spaces with group structure have remarkable algebraic properties (such as cancellation law, existence of center, quasidistributive law, etc.) and can be effectively used for algebraic calculations. Some early results related to quasilinear spaces can be found in [2]–[6], [14].

In this work we study the algebraic properties of intervals in the lines of [11], taking into account that intervals are quasilinear systems. We show that in a quasilinear space with group structure one can introduce analogues of the theory of linear spaces, like linear combinations, linear dependence, basis, etc. The formulation and solution of certain algebraic problems in quasilinear spaces is based on an appropriate terminology and notation. We briefly outline a theory of quasilinear spaces with group structure, which is an extension of the theory of linear spaces. Here we develop the theory using an alternative and more simple approach than the one in [11], considering first in some detail the symmetric case. Only a few proofs are given due to space limit; the reader interested in proofs may consult [11], where the properties of abstract quasilinear spaces are studied and applications to convex bodies and support functions are discussed. The theoretical foundation in this work is developed gradually and step-wise; all omitted proofs are brief and transparent.

Interval operations (for proper intervals). By $\mathbb{R}$ we denote the ordered field of reals; the n-dimensional real vector space is $\mathbb{R}^n$, $n \geq 1$. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, the partial order "$\leq$" is given by $a \leq b \Longleftrightarrow a_i \leq b_i$ for all $i = 1, \ldots, n$. For $a', a'' \in \mathbb{R}^n$, $a'' \geq 0$, the set $A = (a'; a'') = \{a \in \mathbb{R}^n \mid a' - a'' \leq a \leq a' + a''\} = \{a \in \mathbb{R}^n \mid |a' - a| \leq a''\}$ is called an (n-dimensional) *interval* (or *box*) in $\mathbb{R}^n$; a' is the center of A and a'' is the radius of A. The set of all compact intervals in $\mathbb{R}$ is denoted by $I(\mathbb{R})$. The set of all n-dimensional intervals in $\mathbb{R}^n$ is $\mathcal{K} = I(\mathbb{R}^n)$. The set $\mathbb{R}^n$ is considered as a subset of $\mathcal{K}$; as such it is denoted by $\mathcal{K}_D \subset \mathcal{K}$, $\mathcal{K}_D \cong \mathbb{R}^n$ (the letter "D" stands for "distributive" or "degenerate" elements). We have $0 \in \mathcal{K}_D$.

Addition of two intervals is defined by $A + B = \{c \mid c = a + b,\ a \in A,\ b \in B\}$, $A, B \in \mathcal{K}$; here $a, b, c \in \mathbb{R}^n$. The set $\mathcal{K}$ is an abelian cancellative (a. c.) monoid $\mathcal{Q} = (\mathcal{Q}, +)$ with neutral element 0 [1], [8]. We recall that a monoid is a semigroup (i.e. $(A+B)+C = A+(B+C)$) with null (i.e. $A+0 = A$); abelian means commutative (i.e. $(A+B)+C = A+(B+C)$) and cancellative means that the cancellation law (i.e. $A + C = B + C \Longrightarrow A = B$) is satisfied.

Multiplication of an interval by (real) scalar $* : \mathbb{R} \times \mathcal{K} \longrightarrow \mathcal{K}$ is defined by $\alpha * B = \{c \mid c = \alpha b,\ b \in B\}$, $\alpha \in \mathbb{R}$, $B \in \mathcal{K}$, $b, c \in \mathbb{R}^n$. For $A, B, C \in \mathcal{K}$, $\alpha, \beta, \gamma \in \mathbb{R}$, we have:

$$\gamma * (A + B) = \gamma * A + \gamma * B, \tag{1.1}$$
$$\alpha * (\beta * C) = (\alpha\beta) * C, \tag{1.2}$$
$$1 * A = A, \tag{1.3}$$
$$(\alpha + \beta) * C = \alpha * C + \beta * C, \quad \text{if} \ \ \alpha\beta \geq 0. \tag{1.4}$$

Relation (1.4) is called *quasidistributive law*. *Negation* $\neg : \mathcal{K} \longrightarrow \mathcal{K}$ is defined by $\neg A = (-1) * A$. We have $\neg(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (\neg A)$ for $\gamma \in \mathbb{R}$ and $A \in \mathcal{K}$. *Subtraction* is defined by $A \neg B = A + (\neg B)$.

An interval $A \in \mathcal{K}$ is called *(centrally) symmetric*, if $A = \neg A$. The set of all symmetric intervals is $\mathcal{K}_S = \{A \in \mathcal{K} \mid A = \neg A\}$. For $A \in \mathcal{K}$, we have $A \neg A \in \mathcal{K}_S$; indeed, $\neg(A \neg A) = \neg A + A = A \neg A$. The set of all degenerate intervals $A \in \mathcal{K}$ is $\mathcal{K}_D = \{A \in \mathcal{K} \mid A + (\neg A) = 0\} \cong \mathbb{R}^n$. Note that every degenerate interval is *distributive* in the sense that for such intervals relation (1.4) holds for all $\alpha, \beta \in \mathbb{R}$; hence the space $\mathcal{K}_D$ is linear.

Remark 1.1. Instead of "$\neg$" the symbol "$-$" is widely used in the literature on interval (and convex) analysis. Note that $A + (\neg A) \neq 0$ for $A \in \mathcal{K} \setminus \mathcal{K}_D$. Hence, the use of the symbol "$-$" to denote negation may lead to confusion, for "$-$" is normally used to denote opposite elements $\mathrm{opp}(A)$, such that $A + \mathrm{opp}(A) = 0$. We use "$-$" only for elements of a linear space or a field, where opposite and negative coincide and no confusion may occur. We thus avoid the use of "$-$" in the case of intervals.

2. Quasilinear Spaces

Assume that $\mathcal{Q} = (\mathcal{Q}, +)$ is an abelian cancellative (a. c.) monoid with neutral element 0. An element $A \in \mathcal{Q}$ is *invertible*, if there exists $X \in \mathcal{Q}$ such that $A + X = 0$, in this case the (unique) element X is the *opposite* (or *additive inverse*) of A, symbolically $X = \mathrm{opp}(A)$. The set $\mathcal{Q}_I$ of all invertible elements of an a. c. monoid $(\mathcal{Q}, +)$ forms an abelian group $(\mathcal{Q}_I, +)$, which is a nonempty subgroup of the monoid (due to $0 \in Q_I$). In the special case $\mathcal{Q} = \mathcal{Q}_I$ the monoid is a group. A monoid, which is not a group is called a *proper monoid*.

Let $(\mathcal{Q}, +)$ be an a. c. monoid and "$*$" be multiplication by scalar on $\mathbb{R} \times \mathcal{Q}$ satisfying (1.1)–(1.4). The algebraic system $(\mathcal{Q}, +, \mathbb{R}, *)$ is called *quasilinear space (over* $\mathbb{R}$*) with cancellation law* or just *quasilinear space* (in this paper we do not consider quasilinear spaces which are not cancellative). A more general definition of a quasilinear space (not necessarily cancellative) is given in [13].

Multiplication by "-1" is called *negation* in $\mathcal{Q}$ and is denoted by "$\neg$"; we write $\neg B = (-1) * B$; $A \neg B = A + (\neg B)$. Looking at the quasilinear law (1.4) we may ask is there a relation between the two expressions $(\alpha + \beta) * C$ and $\alpha * C + \beta * C$ in the case $\alpha\beta < 0$? The following proposition gives the answer to that question:

PROPOSITION 2.1. *Let $\mathcal{Q}$ be a quasilinear space. Then for all $C \in \mathcal{Q}$, $\alpha, \beta \in \mathbb{R}$,*

$$(\alpha + \beta) * C + \gamma * (C \neg C) = \alpha * C + \beta * C, \tag{2.1}$$

where γ is given by $\gamma = \{0,\ \text{if } \alpha\beta \geq 0;\ \min\{|\alpha|, |\beta|\},\ \text{if } \alpha\beta < 0\}$.

Proof. If $\alpha\beta \geq 0$, then from the quasidistributive law (1.4) we have $\gamma = 0$. Assume $\alpha\beta < 0$. Consider the subcase $\alpha > 0$, $0 < -\beta \leq \alpha$. Denote $\gamma = -\beta > 0$, $\alpha + \beta = \delta \geq 0$ (and hence: $\alpha = \gamma + \delta > 0$). Using $(\gamma + \delta) * C = \gamma * C + \delta * C$, $\gamma\delta \geq 0$, we obtain $\alpha * C + \beta * C = (\gamma + \delta) * C + (-\gamma) * C = \delta * C + \gamma * C + (-\gamma) * C = (\alpha + \beta) * C + \gamma * (C \neg C)$. The remaining cases are treated similarly. □

Obviously, for $\alpha\beta \geq 0$ relations (1.4) and (2.1) are equivalent. Proposition 2.1 implies:

COROLLARY 2.1 (linearity condition) [6], [14]. *For $C \in \mathcal{Q}$ the two conditions: i) $C \neg C = 0$ and ii) $(\alpha + \beta) * C = \alpha * C + \beta * C$ for all $\alpha, \beta \in \mathbb{R}$, are equivalent.*

The set of all invertable elements, whose opposite and negative coincide, is denoted by $\mathcal{Q}_D$, that is, for $A \in \mathcal{Q}_D$ we have $A + (\neg A) = 0$; symbolically $\mathcal{Q}_D = \{A \in \mathcal{Q} \mid A \neg A = 0\}$. A (quasilinear) space $\mathcal{Q}$, such that $\mathcal{Q} = \mathcal{Q}_D$, is linear. The elements of a linear space are said to be *distributive* or *linear*. Note that in a quasilinear space every linear element is invertable ($\mathcal{Q}_D \subset \mathcal{Q}_I$), but the inverse is not necessarily true.

In the case $\mathcal{Q} = \mathcal{Q}_I$ the quasilinear space $\mathcal{Q}$ is an abelian group with respect to addition. A quasilinear space $\mathcal{Q}$ with $\mathcal{Q} = \mathcal{Q}_I$ is called a *quasilinear space with group structure* and will be denoted by $\mathcal{G}$ (in [11] quasilinear spaces with group structure are briefly called "q-linear spaces"). If $\mathcal{G} = \mathcal{G}_D$, then the quasilinear space with group structure $\mathcal{G}$ is linear; hence the practically interesting case is: $\mathcal{G} = \mathcal{G}_I$, $\mathcal{G} \neq \mathcal{G}_D$. Therefore, when saying that a system $\mathcal{G}$ is a quasilinear space with group structure, we shall normally exclude the linear case $\mathcal{G} = \mathcal{G}_D$.

Remark 2.1. As mentioned in the previous remark we use the sign "$-$" only for elements of a linear space (or field), where opposite and negative elements coincide ($A + (\neg A) = 0$ for all elements A), so that no confusion may occur. A quasilinear space with group structure $\mathcal{Q}$ is generally not linear, and we shall use in $\mathcal{Q}$ the notation "opp" instead of "$-$" (unless we speak of some linear subspace of $\mathcal{Q}$).

An element $A \in \mathcal{Q}$, such that $A = \neg A$, is called *(centrally) symmetric*. The set of all symmetric elements of $\mathcal{Q}$ is $\mathcal{Q}_S = \{A \in \mathcal{Q} \mid \neg A = A\}$. The space $\mathcal{Q}_S$ is a subspace of $\mathcal{Q}$. A quasilinear space $\mathcal{Q}$, such that $\mathcal{Q} = \mathcal{Q}_S$, is called *symmetric*. Symmetric quasilinear spaces with group structure present a special interest; if Q is such a space, then we have both $Q = Q_S$ and $Q = Q_I$.

In a quasilinear space $\mathcal{Q}$, if for some $A \in \mathcal{Q}$ the equation $X \neg A = A$ is solvable with respect to X, then the solution X is unique. The element $X / 2 = (1 / 2) * X$, where X is such that $X \neg A = A$, is called the *center* of A and is denoted by $c(A) = X / 2$. Every center is linear (element of the quasilinear space), for summing up the equations $X \neg A = A$ and $\neg X + A = \neg A$ implies $X \neg X = 0$.

A quasilinear space $\mathcal{Q}$, such that $\mathcal{Q} \neq \mathcal{Q}_I$, is a *proper* monoid; such a space is called a *quasilinear space with monoid structure*. Cancellative quasilinear spaces with monoid structure are rather different from quasilinear spaces with group structure and should be studied separately. We shall not follow the unifying approach in [6], [14], and shall introduce basis only in a quasilinear space with group structure.

Every commutative semigroup can be embedded in a group [7], hence every quasilinear space with monoid structure can be embedded in a quasilinear one with group structure; we briefly outline the embedding construction for the special case of an a. c. monoid, resp. a cancellative quasilinear space (with monoid structure). Note that a group is cancellative, hence a quasilinear space with group structure is necessarily cancellative.

Embedding of a quasilinear space with monoid structure into a quasilinear space with group structure. Let $(\mathcal{Q}, +)$ be an a. c. monoid. Denote by $\mathcal{G} = (\mathcal{Q} \times \mathcal{Q}) / \rho = \mathcal{Q}^2 / \rho$ the set of pairs (A, B), $A, B \in \mathcal{Q}$, factorized by the equivalence relation $\rho : (A, B)\rho(C, D) \Longleftrightarrow A + D = B + C$; we shall further write "=" instead of ρ. Denote the equivalence class in $\mathcal{G}$, represented by the pair (A, B), again by (A, B), hence $(A, B) = (A + X, B + X)$. Define addition in

$\mathcal{G}$ by means of: $(A,B) + (C,D) = (A + C,\ B + D)$. The null of $\mathcal{G}$ is the class $(Z,Z) = (0,0) = 0$. The opposite element to $(A,B) \in \mathcal{G}$ is $\mathrm{opp}(A,B) = (B,A)$; indeed, $(A,B) + \mathrm{opp}(A,B) = (A,B) + (B,A) = (A + B,\ B + A) = 0$. The system $(\mathcal{G},+)$ thus obtained is an abelian group. We have $\mathrm{opp}((A,B)+(C,D)) = \mathrm{opp}(A,B)+\mathrm{opp}(C,D)$. The mapping $\varphi : \mathcal{Q} \longrightarrow \mathcal{G}$ defined for $A \in \mathcal{Q}$ by $\varphi(A) = (A,0) \in \mathcal{G}$, is called *embedding*. We *embed* $\mathcal{Q}$ in $\mathcal{G}$ by identifying $A \in \mathcal{Q}$ with the equivalence class $(A,0) = (A + X,\ X)$, $X \in \mathcal{Q}$; all elements of $\mathcal{G}$ admitting the form $(A,0)$ are called *proper*. Hence, the proper elements of $\mathcal{G}$ are all pairs (U,V), $U,V \in \mathcal{Q}$, such that $V + Y = U$ for some $Y \in \mathcal{Q}$, i.e. $(U,V) = (V + Y,\ V) = (Y,0)$. The set of all proper elements of $\mathcal{G}$ is $\varphi(\mathcal{Q}) = \{(A,0) \mid A \in \mathcal{Q}\} \cong \mathcal{Q}$. The set $\mathcal{G}$ is an *extension* of $\mathcal{Q}$; we extend the set $\varphi(\mathcal{Q}) \cong \mathcal{Q}$ of proper elements of the form $(A,0)$ up to the set $\mathcal{G}$ of elements of the form (A,B). Due to $(A,B) = (A,0) + \mathrm{opp}(B,0)$ the pair (A,B) is called the *difference* of A and B and the set $\mathcal{G}$ is called the *difference set* of $\mathcal{Q}$ and denoted $\mathcal{G} = \mathrm{dis}\ \mathcal{Q} = \mathcal{Q}^2 / \rho$; we shall also say that $\mathcal{G}$ is the group generated (induced) by the monoid $\mathcal{Q}$ (using the extension method).

We extend multiplication by scalar $* : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ by the natural definition:

$$\gamma * (A,B) = (\gamma * A, \gamma * B), \quad A,B \in \mathcal{Q}, \ \ \gamma \in \mathbb{R}. \tag{2.2}$$

THEOREM 2.1 [11]. *Let $(\mathcal{Q},+,\mathbb{R},*)$ be a quasilinear space, and $(\mathcal{G},+)$, $\mathcal{G} = \mathrm{dis}\ \mathcal{Q}$, be the induced abelian group of differences. Then the system $(\mathcal{G},+,\mathbb{R},*)$ with "$*$" defined by (2.2) is a quasilinear space with group structure (over $\mathbb{R}$).*

In what follows we shall use lower case Roman letters to denote the elements of a quasilinear space with group structure $\mathcal{G}$, writing e. g. $a = (A', A'') \in \mathcal{G}$. The reason for such a change in notation is that the computational rules in a quasilinear space with monoid structure are rather different from the ones in $\mathcal{G}$—the elements of $\mathcal{G}$ are invertible and thus are close to elements of a linear space (vectors, n-tuples of numbers).

According to Theorem 2.1 for $a,b,c \in \mathcal{G}$, $\alpha,\beta,\gamma \in \mathbb{R}$, we have: $\alpha * (\beta * c) = (\alpha\beta) * c$, $\gamma * (a + b) = \gamma * a + \gamma * b$, $1 * a = a$, and, in addition, the quasidistributive law:

$$(\alpha + \beta) * c = \alpha * c + \beta * c, \quad \alpha\beta \geq 0. \tag{2.3}$$

In the special case when $\mathcal{Q}$ is an abelian group, then $\mathcal{G} = \mathrm{dis}\ \mathcal{Q} \cong \mathcal{Q}$. Hence, the embedding construction makes sense whenever the quasilinear space $\mathcal{Q}$ is a proper monoid—then the induced space $\mathcal{G}$ is an abelian group, and hence is "closer" to a linear space than $\mathcal{Q}$.

Computation in quasilinear spaces with group structure. In the sequel $\mathcal{G}$ denotes a quasilinear space with group structure, such that $\mathcal{G} = \mathcal{G}_I$. Negation in $\mathcal{G}$ is $\neg a = (-1)*a$. We have $\neg(\neg a) = a$, $a \neg b = a + (\neg b) = a + (-1)*b$, $\neg(a+b) = \neg a \neg b$. A symmetric element $a \in \mathcal{G}$ is such that $a = \neg a$; the set of all symmetric elements of $\mathcal{G}$ is $\mathcal{G}_S = \{a \in \mathcal{G} \mid a = \neg a\}$. The set of all linear (distributive) elements of a quasilinear space with group structure $\mathcal{G}$ is $\mathcal{G}_D = \{a \in \mathcal{G} \mid a \neg a = 0\}$. A quasilinear space with group structure consisting only of linear elements ($\mathcal{G} = \mathcal{G}_I = \mathcal{G}_D$) is linear (note that a linear space is a special case of a quasilinear space such that (2.3) holds for $\alpha\beta < 0$ as well); hence the interesting case is $\mathcal{G} = \mathcal{G}_I \neq \mathcal{G}_D$. Elements from $\mathcal{G}_D \cup \mathcal{G}_S$, which are either symmetric or linear, are called *axial*. The null element 0 is the only axial element, which is both symmetric and linear, that is $\mathcal{G}_D \cap \mathcal{G}_S = \{0\}$. The elements from $\mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$, that is the elements which are not axial, will be called *nonaxial*. We shall consider the general case $\mathcal{G} \neq \mathcal{G}_D$, $\mathcal{G} \neq \mathcal{G}_S$, that is $\mathcal{G}$ is not axial; if this is not the case, it will be explicitly mentioned.

The composition of opposite and negation in $\mathcal{G}$ is called *conjugation* or *dualization* and is denoted by a_- (to be read: "a dual" or "a conjugate"), symbolically: $a_- = \mathrm{opp}(\neg a) = \neg(\mathrm{opp}(a))$. Using conjugation we may express opposite as: $\mathrm{opp}(a) = \neg a_- = (-1) * a_-$. For instance, the equality $a = a$ may be equivalently written as $a \neg a_- = 0$. We have $(a_-)_- = a$; also $(a + b)_- = a_- + b_-$. The operators opposite (opp), negation (neg) and conjugate (dual) are discussed in [2], [5].

PROPOSITION 2.2. *For $a \in \mathcal{G}$ we have $a + a_- \in \mathcal{G}_D$, and $a \neg a \in \mathcal{G}_S$. For $a \in \mathcal{G}_S$, $\gamma \in \mathbb{R}$, we have $\gamma * a = (-\gamma) * a = |\gamma| * a$. Also: $a + a_- = 0$, iff $a \in \mathcal{G}_S$; and $a \neg a = 0$, iff $a \in \mathcal{G}_D$. For each $t \in \mathcal{G}_D$ there exists $x \in \mathcal{G}$, such that $t = x + x_-$; all x with this properties are of the form $x = (1/2) * t + s$, where $s \in \mathcal{G}_S$ is arbitrary. We have $\mathcal{G}_D = \{x + x_- \mid x \in \mathcal{G}\}$. For each $s \in \mathcal{G}_S$ there exists $y \in \mathcal{G}$, such that $s = y \neg y$; all y with this properties are of the form $y = (1/2) * s + t$, where $t \in \mathcal{G}_D$ is arbitrary. We have $\mathcal{G}_S = \{y \neg y \mid y \in \mathcal{G}\}$.*

Rules for calculation. Recall that $\mathcal{G}$ is an abelian group. Using the group properties of $\mathcal{G}$ (like: $0 + a = a$, $\mathrm{opp}(a) + a = 0$, $\mathrm{opp}(0) = 0$, $\mathrm{opp}(a + b) = \mathrm{opp}(a) + \mathrm{opp}(b)$, $a + b = a + c \Longrightarrow b = c$, etc.) and Theorem 2.1, one may derive rules for calculation in a quasilinear space with group structure. A list of rules is summarized in the following:

PROPOSITION 2.3. *For all $\alpha, \beta, \gamma \in \mathbb{R}$ and for all $a, b, c \in \mathcal{G}$ the following properties hold true: $\gamma * 0 = 0$, in particular, $(-1) * 0 = \neg 0 = 0$; $0 * a = 0$; $\neg(\gamma * a) = (-\gamma) * a$; $(\alpha - \beta) * c = \alpha * c + (-\beta) * c = \alpha * c \neg \beta * c$, $\alpha\beta \geq 0$; $\gamma * (a \neg b) = \gamma * a \neg \gamma * b$, i.e. $\gamma * (a + (-1) * b) = \gamma * a + (-\gamma) * b$; $\gamma * (a + b_-) = \gamma * a + \gamma * b_-$; $\alpha * c = \beta * c \Longrightarrow \alpha = \beta$ or $c = 0$; $\gamma * a = \gamma * b \Longrightarrow \gamma = 0$ or $a = b$; $\gamma * a = 0 \Longrightarrow \gamma = 0$ or $a = 0$; $a + \mathrm{opp}(a) = a \neg a_- = \neg a + a_- = 0$; $(\alpha * c)_- = \alpha * c_-$; $a + \gamma * b = 0 \Longleftrightarrow a = (-\gamma) * b_- = \neg(\gamma * b_-)$, in particular, $a + b = 0 \Longleftrightarrow a = \neg b_-$.*

Special notation. For $a \in \mathcal{G}$ denote $a_+ = a$; then for $\lambda \in \{+, -\}$ the element $a_\lambda \in \mathcal{G}$ (read: "a dualized by λ" or "a conjugated by λ") is either a or a_- according to the binary value of λ. Using this notation we can write $(a + b)_\lambda = a_\lambda + b_\lambda$. The equalities $a = b$ and $a_\lambda = b_\lambda$ are algebraically equivalent; each one is obtained from the other via conjugation (dualization) by λ. We have $(a_\mu + b_\nu)_\lambda = a_{\lambda\mu} + b_{\lambda\nu}$, in particular, $(a + b)_\lambda = a_\lambda + b_\lambda$; $(\alpha * c_\mu)_\nu = \alpha * c_{(\mu\nu)}$, in particular, $(\alpha * c_\mu)_\mu = \alpha * c$,

$(a_\lambda)_\mu = a_{\lambda\mu}$, where products of the form $\lambda\mu$ are computed according to the sign rules: $++ = -- = +$, $+- = -+ = -$.

It is important to know how the quasidistributive law looks like in $\mathcal{G}$ in the case $\alpha\beta < 0$. Of course, relation (2.1) is valid in $\mathcal{G}$, but the next result shows a more useful relation.

THEOREM 2.2 [11]. *For $c \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$:*

$$(\alpha+\beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}, \tag{2.4}$$

$$(\alpha+\beta) * c = \alpha * c_{\sigma(\alpha)\sigma(\alpha+\beta)} + \beta * c_{\sigma(\beta)\sigma(\alpha+\beta)}, \tag{2.5}$$

where $\sigma(\alpha) = \{+,\ \textit{if}\ \alpha \geq 0;\ -,\ \textit{if}\ \alpha < 0\}$.

It can be shown that in a quasilinear space with group structure relations (2.4), (2.5) are equivalent to (2.3). The use of symbolic dualization makes formulae (2.4), (2.5) very convenient for symbolic computations. For comparison, without the use of the symbol "c_λ", formula (2.5) obtains the following form, which requires a knowledge of $\sigma(\alpha), \sigma(\beta)$, and, in some cases, of $\sigma(\alpha+\beta)$, and hence, is not appropriate for symbolic manipulations:

$$(\alpha+\beta) * c = \begin{cases} \alpha * c + \beta * c, & \text{if } \sigma(\alpha) = \sigma(\beta) \ \ (= \sigma(\alpha+\beta)), \\ \alpha * c + \beta * c_-, & \text{if } \sigma(\alpha) = -\sigma(\beta), \quad \sigma(\alpha+\beta) = \sigma(\alpha), \\ \alpha * c_- + \beta * c, & \text{if } \sigma(\alpha) = -\sigma(\beta), \quad \sigma(\alpha+\beta) = \sigma(\beta). \end{cases}$$

3. Algebraic Transformations and Quasilinear Equations

Relation (2.4) implies that an expression $\alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)}$ can be transformed to $(\alpha+\beta) * x_{\sigma(\alpha+\beta)}$ for all $x \in \mathcal{G}$ and all $\alpha, \beta \in \mathbb{R}$. Using such transformation we can easily solve an equation for x of the form $\alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)} = d$. Consider now the expression

$$\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)}. \tag{3.1}$$

PROPOSITION 3.1. *If x is axial ($x \in \mathcal{G}_D \cup \mathcal{G}_S$), then (3.1) reduces to $\gamma * x_{\sigma(\gamma)} \in \mathcal{G}_D \cup \mathcal{G}_S$, $\gamma = \alpha \pm \beta$. For $x \in \mathcal{G}_D$ we have: $\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = \gamma * x_{\sigma(\gamma)} = \gamma * x_{-\sigma(\gamma)} = \gamma * x$ with $\gamma = \alpha + \beta$. For $x \in \mathcal{G}_S$ we obtain $\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = \alpha * x_{\sigma(\alpha)} + (-\beta) * x_{-\sigma(\beta)} = \gamma * x_{\sigma(\gamma)} = |\gamma| * x_{\sigma(\gamma)}$, where $\gamma = \alpha - \beta$.*

Proposition 3.1 deals with possible simplification of (3.1). We see that x can be factored out if x is axial. In general, one can write formulae in the lines of (2.1), e. g. for all $c \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$,

$$(\alpha+\beta) * c_{\sigma(\alpha+\beta)} + \alpha * (c \neg c)_{-\sigma(\alpha)} = \alpha * c_{-\sigma(\alpha)} + \beta * c_{\sigma(\beta)},$$

$$(\alpha+\beta) * c_{\sigma(\alpha+\beta)} + \beta * (c \neg c)_{-\sigma(\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)},$$

or, equivalently, for all $c \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$,

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} + \gamma * (c \neg c)_{-\sigma(\gamma)} = \alpha * c_{\lambda\sigma(\alpha)} + \beta * c_{\sigma(\beta)}, \tag{3.2}$$

where γ is given by $\gamma = \{0, \text{ if } \lambda = +; \ \alpha, \text{ if } \lambda = -\}$.

Minding that x cannot be simply factored out in (3.1), the first impression is that one cannot solve an equation of the form

$$\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = d \tag{3.3}$$

with respect to x in general. However, the next Proposition 3.2 shows that equation (3.3) is solvable although the left-hand side cannot be reduced to one term.

PROPOSITION 3.2. *Let $\alpha, \beta \in \mathbb{R}$, $d \in \mathcal{G}$. If $\gamma = \alpha^2 - \beta^2 \neq 0$, then equation (3.3) is equivalent to*

$$x = \gamma^{-1} * (\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)})_{\sigma(\gamma)}, \tag{3.4}$$

which is the unique solution of (3.3). If $\beta = \alpha \neq 0$, then (3.3) is equivalent to $x + x_- = \alpha^{-1} * d_{\sigma(\alpha)}$ and, hence, is solvable iff $d \in \mathcal{G}_D$; we have $x = (1/2) * d + s$, $s \in \mathcal{G}_S$. If $\beta = -\alpha \neq 0$, then (3.3) is equivalent to $x \neg x = \alpha^{-1} * d_{\sigma(\alpha)}$ and is solvable iff $d \in \mathcal{G}_S$; we have $x = (1/2) * d + t$, $t \in \mathcal{G}_D$.

Note that the expression $\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)} = \alpha * d_{\sigma(\alpha)} + (-\beta) * d_{-\sigma(-\beta)}$ in the right-hand side of the solution (3.4) is of the form $\alpha * d_{\sigma(\alpha)} + \beta' * d_{-\sigma(\beta')}$, $\beta' = -\beta$, that is in the form (3.1). The following propositions hold true [11].

PROPOSITION 3.3. *Assume that $\alpha^2 - \beta^2 \neq 0$, then the equation for $c \in \mathcal{G}$*

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0 \tag{3.5}$$

implies $c = 0$ (that is, $c = 0$ is the unique solution to (3.5)).

PROPOSITION 3.4. *If $(\alpha, \beta) \neq (0, 0)$, then (3.5) implies $c \in \mathcal{G}_D \cup \mathcal{G}_S$.*

PROPOSITION 3.5. *Assume that $c \in \mathcal{G}$, $c \neq 0$ and equation (3.5) holds. Then $\alpha^2 - \beta^2 = 0$. Furthermore, if $\alpha \neq 0$ (and hence $\beta \neq 0$), then $c \in (\mathcal{G}_D \cup \mathcal{G}_S) \setminus \{0\}$.*

PROPOSITION 3.6. *Let $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. Then (3.5) implies $\alpha = \beta = 0$.*

Subspaces. A nonempty subset $\mathcal{H}$ of $\mathcal{G}$ is a *subspace* of $\mathcal{G}$, if $\mathcal{H}$ is a subgroup of $\mathcal{G}$ under addition (inherited from $\mathcal{G}$) and is closed under multiplication by scalar (inherited from $\mathcal{G}$). In more detail, $\mathcal{H}$ is a subspace of $\mathcal{G}$ if and only if $\mathcal{H} \subset \mathcal{G}$ and: i) $-a \in \mathcal{H}$ and $a + b \in \mathcal{H}$ for all $a, b \in \mathcal{H}$ (i.e. $\mathcal{H}$ is a subgroup of $\mathcal{G}$ under addition); ii) $\alpha * c \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$, $c \in \mathcal{H}$ (i.e. $\mathcal{H}$ is closed under multiplication by scalar). The following subspace criterion takes place:

LEMMA 3.1. *$\mathcal{H}$ is a subspace of the quasilinear space with group structure $\mathcal{G}$, if and only if $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}$ is closed under "+", "*", "dual", i.e.: i) $a + b \in \mathcal{H}$ for all $a, b \in \mathcal{H}$; ii) $\alpha * c \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$, $c \in \mathcal{H}$; iii) $a_- \in \mathcal{H}$ for all $a \in \mathcal{H}$.*

It is easy to check that G_D and $\mathcal{G}_S$ are subspaces of $\mathcal{G}$; we call them the linear subspace and the symmetric (quasilinear) subspace, respectively.

For $c \in \mathcal{G}$ the set $\operatorname{span}(c) = \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\}$ is a subspace of $\mathcal{G}$. Every element of span(c) has a unique representation in the form (3.1) if and only if $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. The set span($c$) is said to be spanned by c. This space is the intersection of subspaces of $\mathcal{G}$ containing c.

Direct sum. Sum and direct sum of quasilinear spaces with group structure is defined straightforward as in linear spaces. Namely, for two quasilinear spaces with group structure $\mathcal{G}$, $\mathcal{H}$ there is a least subspace containing both $\mathcal{G}$ and $\mathcal{H}$, called their sum and written $\mathcal{G} + \mathcal{H}$. We have $\mathcal{G} + \mathcal{H} = \{u + v \mid u \in \mathcal{G}, v \in \mathcal{H}\}$. Similarly, one defines the sum of any finite set of subspaces $\mathcal{G}_1, \ldots, \mathcal{G}_k$: $\mathcal{G}_1 + \cdots + \mathcal{G}_k = \{\sum u_i \mid u_i \in \mathcal{G}_i\}$. Let $\mathcal{G}$ be a quasilinear space with group structure and $\mathcal{H}_1, \ldots, \mathcal{H}_k$ any subspaces of $\mathcal{G}$. We say that $\mathcal{G}$ is the *direct sum* of the spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$ and write $\mathcal{G} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$, if each $u \in \mathcal{G}$ can be uniquely expressed in the form $u = v_1 + v_2 + \cdots + v_k$, where $v_i \in \mathcal{H}_i$, $i = 1, \ldots, k$. One can show: 1) A sum $\mathcal{G} + \mathcal{H}$ is direct if $u_1 + v_1 = u_2 + v_2$, $u_1, u_2 \in \mathcal{G}$, $v_1, v_2 \in \mathcal{H}$ imply $u_1 = u_2$, $v_1 = v_2$ (or, equivalenly, $u + v = 0$, $u \in \mathcal{G}$, $v \in \mathcal{H}$ imply $u = 0$, $v = 0$); 2) $\mathcal{Q} = \mathcal{G} \oplus \mathcal{H} \Longleftrightarrow \mathcal{Q} = \mathcal{G} + \mathcal{H}$ and $\mathcal{G} \cap \mathcal{H} = 0$.

4. Symmetric Quasilinear Spaces with Group Structure

We shall assume in this section that the quasilinear space with group structure $\mathcal{G}$ is symmetric, $\mathcal{G} = \mathcal{G}_S$. From Proposition 3.1 we know that for $x \in \mathcal{G} = \mathcal{G}_S$, expression (3.1) reduces to the form $\gamma * x_{\sigma(\gamma)} \in \mathcal{G}_D \cup \mathcal{G}_S$, $\gamma \in \mathbb{R}$. This suggests that in a symmetric quasilinear space with group structure we may define linear combination, linear dependence and basis as follows.

DEFINITION 4.1. Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathcal{G}$ ($\mathcal{G} = \mathcal{G}_S$). An element g of $\mathcal{G}$ of the form

$$g = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}, \tag{4.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, is called a *linear combination* of $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$.

A linear combination (4.1) having a trivial system $(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0)$ is called *trivial*. We write symbolically (4.1) as $g = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)}$.

PROPOSITION 4.1. *Let $h^{(1)}, h^{(2)}, \ldots, h^{(l)}$ be l linear combinations of $c^{(1)}, c^{(2)}, \ldots,$ $c^{(k)} \in \mathcal{G} = \mathcal{G}_S$ and g be a linear combination of $h^{(1)}, h^{(2)}, \ldots, h^{(l)}$. Then g is a linear combination of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$.*

PROPOSITION 4.2. *Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G} = \mathcal{G}_S$, $k \geq 1$. The set $\mathcal{H} = \left\{ \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha i)} \mid \alpha_i \in \mathbb{R} \right\}$ of all linear combinations of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ is a subspace of $\mathcal{G}$.*

We say that the elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G} = \mathcal{G}_S$ form a *generating set* of the subspace $\mathcal{H}$ defined in Proposition 4.2 or that $\mathcal{H}$ is *spanned* by $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ and write $\mathcal{H} = \mathrm{span}\{c^{(1)}, c^{(2)}, \ldots, c^{(k)}\}$. The space $\mathcal{H}$ is the intersection of all subspaces of $\mathcal{G}$ containing $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$.

DEFINITION 4.2. Let $\mathcal{G} = \mathcal{G}_S$ be a symmetric quasilinear space with group structure over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, $k \geq 1$, are *linearly dependent (over* $\mathbb{R}$*)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i.e. if there exist a nontrivial system $\{\alpha_i\}_{i=1}^k$, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{4.2}$$

Elements of $\mathcal{G}$, which are not linearly dependent, are *linearly independent*. In other words the elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$ are *linearly independent*, if (4.2) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, \ldots, k$. Clearly, a single element $c \in \mathcal{G}$ is linearly dependent, iff $c = 0$, and is linearly independent, iff $c \neq 0$.

PROPOSITION 4.3. *Let $\mathcal{G} = \mathcal{G}_S$ be a symmetric quasilinear space with group structure over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, $k \geq 2$, are linearly dependent, iff at least one of the elements is a linear combination of the other elements.*

Similarly, given k elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$, if $l < k$ elements of them are linearly dependent, then all k elements are also linearly dependent. In particular, if $c^{(i)} = 0$ for some i, $1 \leq i \leq k$, then the elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ are linearly dependent.

DEFINITION 4.3. Let $\mathcal{G} = \mathcal{G}_S$ be a symmetric quasilinear space with group structure over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a *basis* of $\mathcal{G}$, if $c^{(i)}$ are linearly independent and $\mathcal{G} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$.

THEOREM 4.1. *Let $\mathcal{G} = \mathcal{G}_S$ be a symmetric quasilinear space with group structure over $\mathbb{R}$. A set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a basis of $\mathcal{G}$, iff every $g \in \mathcal{G}$ can be presented in the form (4.1) in a unique way (i.e. with unique scalars α_i).*

Theorem 4.1 implies that, if $\{c^{(i)}\}_{i=1}^k$ is a basis of $\mathcal{G}$, i.e. $\mathcal{G} = \mathrm{span}\{c^{(1)}, c^{(2)}, \ldots, c^{(k)}\}$, then every $\mathcal{G}_i = \mathrm{span}\{c^{(i)}\}$ is a quasilinear subspace of $\mathcal{G}$; moreover we have $\mathcal{G}_i \cap \mathcal{G}_j = 0$, $i \neq j$. Therefore $\mathcal{G}$ is a *direct sum* of $\mathcal{G}_i$: $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_k = \mathrm{span}\{c^{(1)}\} \oplus \mathrm{span}\{c^{(2)}\} \oplus \cdots \oplus \mathrm{span}\{c^{(k)}\}$. It can be proved that if $\mathcal{G}$ is spanned over a basis of k elements, then any other basis of $\mathcal{G}$ consists of k elements. The number k is called the *dimension* of $\mathcal{G}$.

If $\{c^{(i)}\}$ is a fixed basis of $\mathcal{G}$ then the reals α_i in (4.1), presenting uniquely $f \in \mathcal{G}$ are called *(symmetric) coordinates* of g. The correspondence $g \longrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a bijection, so that we can write $g = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. The arithmetic operations in $\mathcal{G}$ induce corresponding operations in $\mathbb{R}^k$ as follows:

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) + (\beta_1, \beta_2, \ldots, \beta_k) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k), \tag{4.3}$$

$$\gamma * (\alpha_1, \alpha_2, \ldots, \alpha_k) = (|\gamma|\alpha_1, |\gamma|\alpha_2, \ldots, |\gamma|\alpha_k). \tag{4.4}$$

To prove (4.4) we note that $\gamma * g = \gamma * (\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}) = ((\gamma\,\alpha_1) * c^{(1)}_{\sigma(\gamma\alpha_1)} + (\gamma\alpha_2) * c^{(2)}_{\sigma(\gamma\alpha_2)} + \cdots + (\gamma\alpha_k) * c^{(k)}_{\sigma(\gamma\alpha_k)})_{\sigma(\gamma)} = (\gamma\alpha_1, \gamma\alpha_2, \ldots, \gamma\alpha_k)_{\sigma(\gamma)} = (|\gamma|\alpha_1, |\gamma|\alpha_2, \ldots, |\gamma|\alpha_k)$. Clearly, the system $(\mathbb{R}^k, +, \mathbb{R}, *)$ with addition (4.3) and multiplication by real scalar (4.4) is a symmetric quasilinear space (with group structure) which is isomorphic to $(\mathcal{G}, +, \mathbb{R}, *)$. Negation in $\mathcal{G} = \mathcal{G}_S$ is same as identity. Conjugation in $\mathcal{G} = \mathcal{G}_S$ coincides with opposite: $\mathrm{opp}(a) = a_- = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{-\sigma(\alpha_i)} = \sum_{i=1}^{k} (-\alpha_i) * c^{(i)}_{\sigma(-\alpha_i)}$. This implies in terms of $(\mathbb{R}^k, +, \mathbb{R}, *)$:

$$(\alpha_1, \alpha_2, \ldots, \alpha_k)_- = -(\alpha_1, \alpha_2, \ldots, \alpha_k) = (-\alpha_1, -\alpha_2, \ldots, -\alpha_k). \tag{4.5}$$

Linear multiplication in $\mathcal{G}$. In $(\mathbb{R}^k, +, \mathbb{R}, *)$ we introduce the operation "$\cdot$": $\mathbb{R} \times R^k \longrightarrow \mathbb{R}^k$ by means of $\lambda \cdot x = \lambda * x_{\sigma(\lambda)}$. Clearly, the system $(\mathbb{R}^k, +, \mathbb{R}, \cdot)$ is a linear space. However, this space does not contain the operation "$*$", which is important for our purposes. Conversely, the quasilinear space $(\mathbb{R}^k, +, \mathbb{R}, *)$ possesses the linear multiplication "$\cdot$" (the dot may be omitted). Using this operation we may write: $\lambda * x = \lambda x_{\sigma(\lambda)} = \lambda\sigma(\lambda)x = |\lambda|x$.

Equations. The general form of an equation involving one variable in $\mathcal{G} = \mathcal{G}_S$ is $\alpha * x = c$, having a solution $x = \alpha^{-1} * c$. We demonstrate the coordinate method on the same example. Let $b \in \mathcal{G}$, $b \neq 0$, be an arbitrary basis in $\mathcal{G} = \mathcal{G}_S$, assume that $c = \gamma * b_{\sigma(\gamma)}$ and denote $x = \xi * b_{\sigma(\xi)}$. Substituting in $\alpha * x = c$, we obtain the equation $(\alpha\xi) * b_{\sigma(\xi)} + (-\gamma) * b_{-\sigma(\gamma)} = 0$. If $\alpha \geq 0$, then, by the quasidistributive law, this equation reduces to $(\alpha\xi) * b_{\sigma(\alpha\xi)} + (-\gamma) * b_{\sigma(-\gamma)} = (\alpha\xi - \gamma) * b_{\sigma(\alpha\xi - \gamma)} = 0$, which is equivalent to $\alpha\xi - \gamma = 0$, resp. $\xi = \gamma / \alpha$. If $\alpha < 0$, then we have $(\alpha\xi) * b_{-\sigma(\alpha\xi)} + (-\gamma) * b_{-\sigma(\gamma)} = 0$; using Proposition 3.1 we obtain $\alpha\xi + \gamma = 0$, resp. $\xi = -\gamma / \alpha$. Summarizing both cases we obtain $\xi = \gamma / |\alpha|$. This is a special case of the general case of a system of linear equations, see e.g. [9].

5. Nonaxial Quasilinear Spaces with Group Structure

Basis in nonaxial quasilinear spaces with group structure. In this section we assume that $\mathcal{G}$ is a quasilinear space with group structure, which is not axial, that is not linear ($\mathcal{G} \neq \mathcal{G}_D$), and not symmetric ($\mathcal{G} \neq \mathcal{G}_S$). In $\mathcal{G}$ we define (nonaxial) basis as follows.

DEFINITION 5.1. Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathcal{G}$. An element g of $\mathcal{G}$ of the form

$$g = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \beta_1 * c^{(1)}_{-\sigma(\beta_1)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} + \beta_k * c^{(k)}_{-\sigma(\beta_k)}, \tag{5.1}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k \in \mathbb{R}$, is called a *linear combination* of $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$.

PROPOSITION 5.1. *Let $d^{(1)}, d^{(2)}, \ldots, d^{(l)}$ be l linear combinations of $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$ and g be a linear combination of $d^{(1)}, d^{(2)}, \ldots, d^{(l)}$. Then g is a linear combination of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$.*

PROPOSITION 5.2. *Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, $k \geq 1$. The set $\mathcal{H} = \left\{ \sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) \mid \alpha_i, \beta_i \in \mathbb{R} \right\}$ of all linear combinations of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ is a subspace of $\mathcal{G}$.*

DEFINITION 5.2. We say that the subspace $\mathcal{H}$ defined in Proposition 5.2 is *spanned* by $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$ and write $\mathcal{H} = \mathrm{span}\{c^{(1)}, c^{(2)}, \ldots, c^{(k)}\}$. The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ are a *generating set* for $\mathcal{H}$. The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, $k \geq 1$, are *linearly dependent (in $\mathcal{G}$ over $\mathbb{R}$)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i.e. if there exists a nontrivial system $\{(\alpha_i, \beta_i)\}_{i=1}^{k}$, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \beta_1 * c^{(1)}_{-\sigma(\beta_1)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} + \beta_k * c^{(k)}_{-\sigma(\beta_k)} = 0. \tag{5.2}$$

The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$ are *linearly independent*, if (5.2) is possible only for the trivial linear combination, such that $\alpha_i = \beta_i = 0$ for all $i = 1, \ldots, k$. Using Propositions 3.4 and 3.6 we conclude that a single element of $\mathcal{G}$ is linearly dependent iff it is axial, and is linearly independent iff nonaxial.

PROPOSITION 5.3. *The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, $k \geq 2$, are linearly dependent, iff at least one of the elements is a linear combination of the other elements.*

Given k elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathcal{G}$, if $l < k$ elements of them are linearly dependent, then all k elements are also linearly dependent. In particular, if $c^{(i)}$ for some i, $1 \leq i \leq k$, is axial, then $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ are linearly dependent.

DEFINITION 5.3. The set $\{c^{(i)}\}_{i=1}^{k}$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a *(nonaxial) basis* of $\mathcal{G}$, if $c^{(i)}$ are linearly independent and $\mathcal{G} = \mathrm{span}\{c^{(i)}\}_{i=1}^{k}$.

THEOREM 5.1. *A set $\{c^{(i)}\}_{i=1}^{k}$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a basis of $\mathcal{G}$, iff every $g \in \mathcal{G}$ can be presented in the form (5.1) in a unique way (i.e. with unique scalars α_i, β_j).*

According to Theorem 5.1, if $\{c^{(i)}\}_{i=1}^{k}$ is a basis of $\mathcal{G}$, then $\mathcal{G}_i = \mathrm{span}\{c^{(i)}\}$ is a quasilinear subspace of $\mathcal{G}$ for any $i = 1, \ldots, k$; moreover, we have $\mathcal{G}_i \cap \mathcal{G}_j = 0$, $i \neq j$. Hence we can present $\mathcal{G}$ as a *direct sum* of $\mathcal{G}_i$: $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_k$, that is $\mathrm{span}\{c^{(1)}, c^{(2)}, \ldots, c^{(k)}\} = \mathrm{span}\{c^{(1)}\} \oplus \cdots \oplus \mathrm{span}\{c^{(k)}\}$. In a quasilinear space with a nonaxial basis the symmetric dimension is equal to the linear one, $k = m$; such is the case with m-dimensional intervals.

Transition from nonaxial to axial basis. Recall that if $\mathcal{G}$ is nonaxial, then a single element $c \in \mathcal{G}$ is linearly dependent if and only if c is axial ($c \in \mathcal{G}_D \cup \mathcal{G}_S$); in particular, the element $c = 0$ is linearly dependent. Indeed, consider equation (3.5) for c: $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0$ with $(\alpha, \beta) \neq (0, 0)$, that is assume that the linear combination of c is nontrivial. From Proposition 3.4 we know that there exists a solution $c \in \mathcal{G}_D \cup \mathcal{G}_S$ to (3.5).

PROPOSITION 5.4. *If c is nonaxial, i.e. $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$, then $d = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}$ is axial ($d \in \mathcal{G}_D \cup \mathcal{G}_S$), if and only if $\alpha^2 - \beta^2 = 0$. More specifically:*

i) d is linear, i.e. $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_D$, iff $\alpha = \beta$;

ii) d is symmetric, i.e. $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_S$, iff $\alpha = -\beta$.

PROPOSITION 5.5. *Assume $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\mathcal{H} = \mathrm{span}\{c\}$. Then $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_D$.*

Proof. By assumption $\mathcal{H} = \mathrm{span}\{c\} = \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\}$. For the symmetric, resp. linear, subspace of $\mathcal{H}$ we have:

$$\begin{aligned}
\mathcal{H}_S &= \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha = -\beta\} = \{\alpha * (c \neg c)_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\},\\
\mathcal{H}_D &= \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha = \beta\} = \{\alpha * (c + c_-) \mid \alpha \in \mathbb{R}\}.
\end{aligned}$$

Every element $c \in \mathcal{H}$ can be presented as $c = s + t$, where $s = (c \neg c) / 2 \in \mathcal{H}_S$, $t = (c + c_-) / 2 \in \mathcal{H}_D$, hence, $\mathcal{H} = \mathcal{H}_S + \mathcal{H}_D$. More specifically, we have

$$\mathcal{H} = \{\alpha * s_{\sigma(\alpha)} + \beta * t_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\} = \{|\alpha| * s_{\sigma(\alpha)} + \beta * t \mid \alpha, \beta \in \mathbb{R}\}. \quad (5.3)$$

Due to $\mathcal{H}_S \cap \mathcal{H}_D = 0$ and $\mathcal{H}_S = \mathrm{span}\{s\} = \{\alpha * s_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\} = \{|\alpha| * s_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\}$; $\mathcal{H}_D = \mathrm{span}\{t\} = \{\beta * t \mid \beta \in \mathbb{R}\}$, we see that $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_D$. □

The elements $s = s(c) = (c \neg c) / 2 \in \mathcal{H}_S$, $t = t(c) = (c + c_-) / 2 \in \mathcal{H}_D$ are *projections* of c on the *axial* subspaces $\mathcal{H}_S$, resp. $\mathcal{H}_D$. If $c \in \mathcal{G}$ is fixed, then $s(c)$ and $t(c)$ are also fixed. The coefficients α, β in (5.3) are the *(center-radius) coordinates* of the element $g = \alpha * s_{\sigma(\alpha)} + \beta * t_{-\sigma(\beta)}$; α is the *radius* and β is the *center* of g. Clearly, s is a basis in $\mathcal{H}_S$, and t is a basis in $\mathcal{H}_D$.

Proposition 5.5 can be generalized for a nonaxial quasilinear space with group structure $\mathcal{G}$ spanned over a (nonaxial) generating set of k elements $c^{(i)} \in \mathcal{G} \setminus (\mathcal{G}_S \cup \mathcal{G}_D)$, $i = 1, \ldots, k$. Consider the corresponding projections on the axial spaces, that is the k symmetric elements $s^{(i)} = (c^{(i)} \neg c^{(i)}) / 2 \in \mathcal{G}_S$, and the k linear elements $t^{(i)} = (c^{(i)} + c_-^{(i)}) / 2 \in \mathcal{G}_D$, such that $c^{(i)} = s^{(i)} + t^{(i)}$, $i = 1, \ldots, k$. Hence a generating set (basis) in $\mathcal{G}$ induces respective generating sets (bases) in the two axial subspaces (the symmetric and the linear one), so that we can present the elements of $\mathcal{G}$ uniquely in the form $\sum_{i=1}^{k} (|\alpha_i| * s^{(i)}_{\sigma(\alpha_i)} + \beta_i * t^{(i)})$. Thus the following generalization of Proposition 5.5 holds:

THEOREM 5.2. *Let $\mathcal{G}$ be a nonaxial quasilinear space with group structure with a finite basis $c^{(1)}, \ldots, c^{(k)}$. Then $\mathcal{G} = \mathcal{G}_S \oplus \mathcal{G}_D$. Moreover, the system $s^{(i)} = (c^{(i)} \neg c^{(i)}) / 2$, $i = 1, \ldots, k$, is a basis for the symmetric subspace $\mathcal{G}_S$, and the system $t^{(i)} = (c^{(i)} + c_-^{(i)}) / 2$, $i = 1, \ldots, k$ is a basis for the linear susbspace $\mathcal{G}_D$.*

If $\{c^{(i)}\}$ is a nonaxial basis of $\mathcal{G}$, then $\{s^{(i)}\}$ is a (symmetric) basis of $\mathcal{G}_S$ and $\{t^{(i)}\}$ is a (linear) basis of $\mathcal{G}_D$; the dimensions of the symmetric space $\mathcal{G}_S$ and linear space $\mathcal{G}_D$ are equal.

The one-dimensional interval case. Consider the monoid of all (proper) one-dimensional intervals $(I(\mathbb{R}), +)$. Intervals are usually presented in end-point form, however the above theory of quasilinear spaces points out that the natural presentation of intervals is in center-radius form. That center-radius coordinates are most suitable for computations with intervals has been practically motivated by T. Sunaga, see [12]. Introducing interval multiplication by scalar we obtain the system $(I(\mathbb{R}), +, \mathbb{R}, *)$, which is a cancellative quasilinear space with monoid structure.

The space $(\mathcal{I}, +, \mathbb{R}, *)$, $\mathcal{I} = \mathrm{dis}I(\mathbb{R}) = I(\mathbb{R})^2 / \rho$, is the induced quasilinear space with group structure of extended (generalized, Kaucher, directed, modal etc.) intervals [2], [5], [10]; $\mathcal{I}$ is a direct sum of a one-dimensional linear space (of the centers of the intervals) and a one-dimensional symmetric quasilinear space with group structure (of generalized symmetric intervals), $\mathcal{I} = \mathcal{I}_D \oplus \mathcal{I}_S$, $\mathcal{I}_D \cong \mathbb{R}$.

Let t be a basis of $\mathcal{I}_D$, so that the coordinates of t in $\mathcal{I} = \mathcal{I}_D \oplus \mathcal{I}_S$ are 1, resp. 0; we simply write $t = (1; 0)$; also let s be the basis of $\mathcal{I}_S$ so that $s = (0; 1)$, so that the pair t, s is a basis of $\mathcal{I}$. Denoting the center/radius coordinates of $a \in \mathcal{I}$ by a', resp. a'', we write $a = (a'; a'') \in \mathcal{I}$ with $a', a'' \in \mathbb{R}$. We thus have: $a = a' * t + a'' * s_{\sigma(a'')}$. Some operations in center-radius form are: $(a'; a'') + (b'; b'') = (a' + b'; a'' + b'')$, $\gamma * (a'; a'') = (\gamma a'; |\gamma| a'')$, $\neg(a'; a'') = (-a'; a'')$, $\mathrm{opp}(a'; a'') = (-a'; -a'')$, $(a'; a'')_- = (a'; -a'')$, $(a'; a'') \neg (b'; b'') = (a' - b'; a'' + b'')$, $(a'; a'') + (b'; b'')_- = (a' + b'; a'' - b'')$, $(a'; a'') \neg (b'; b'')_- = (a' - b'; a'' - b'')$, $\gamma * (a'; a'')_- = (\gamma a'; -|\gamma| a'')$, $\gamma * (a'; a'')_{\sigma(\gamma)} = (\gamma a'; \gamma a'')$. The last formula presents the linear multiplication by scalar $\gamma \cdot (a'; a'') = (\gamma a'; \gamma a'') = \gamma * (a'; a'')_{\sigma(\gamma)}$ [3]. An interpretation of this operation and some applications are given in [11].

Concluding remarks. We have shown that in an interval quasilinear space with group structure one can naturally extend concepts from linear algebra, such as linear combination, linear dependence and basis. These concepts allow to formulate and solve certain classes of interval algebraic problems in the framework of linear algebra. In particular interval algebraic problems can be reformulated as real algebraic problems for the coordinates of the intervals without increasing the dimension of the problem [9]. For the interpretation of solutions the theory of modal intervals can be used [4]. A simple interpretation can be obtained by a presentation of the interval quasilinear space with group structure as an union of two quasilinear spaces with monoid structure similar to the decomposition of the system of real numbers into two quasilinear systems with monoid structures: one of nonnegative and one of nonpositive numbers. Such a presentation for one-dimensional intervals has been studied abstractly in [10].

Acknowledgements

This article arose from a lecture presented at the MISC'99 workshop in Girona, Spain, held in February 1999; the author thanks the organizers of the meeting for their invitation and support. This work was partially supported by the Bulgarian National Science Fund under grant No. I-903/99.

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